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On Generalized Hermite–Fejér Interpolation of Lagrange Type on the Chebyshev Nodes

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For $f \in C[-1, 1]$, let $H_{m,n}(f, x)$ denote the $(0, 1, ..., m)$ Hermite–Fejér (HF) interpolation polynomial of f based on the Chebyshev nodes. That is, $H_{m,n}(f, x)$ is the polynomial of least degree which interpolates $f(x)$ and has its first m derivatives vanish at each of the zeros of the nth Chebyshev polynomial of the first kind. In this paper a precise pointwise estimate for the approximation error $|H_{2m,n}(f, x) - f(x)|$ is developed, and an equiconvergence result for Lagrange and $(0, 1, ..., 2m)$ HF interpolation on the Chebyshev nodes is obtained. This equiconvergence result is then used to show that a rational interpolatory process, obtained by combining the divergent Lagrange and $(0, 1, ..., 2m)$ HF interpolation methods on the Chebyshev nodes, is convergent for all $f \in C[-1, 1]$.

1. INTRODUCTION

Suppose

$$X = \{x_{k,n} : k = 1, 2, ..., n;\ n = 1, 2, 3, ...\}$$

is a triangular matrix such that, for all n,

$$1 \geqslant x_{1,n} > x_{2,n} > \cdots > x_{n,n} \geqslant -1,$$

and let f be a real-valued function defined on the interval $[-1, 1]$. Then, for each integer $m \geqslant 0$, there exists a unique polynomial $H_{m,n}(X, f, x)$ of degree at most $(m+1)\,n - 1$ which satisfies

$$\begin{cases} H_{m,n}(X, f, x_{k,n}) = f(x_{k,n}), & 1 \leqslant k \leqslant n, \\ H^{(r)}_{m,n}(X, f, x_{k,n}) = 0, & 1 \leqslant r \leqslant m,\ 1 \leqslant k \leqslant n. \end{cases}$$

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$H_{m,n}(X, f, x)$ is referred to as the $(0, 1, ..., m)$ Hermite–Fejér (HF) interpolation polynomial of $f(x)$. Observe that $H_{0,n}(X, f, x)$ is the well-known Lagrange interpolation polynomial of $f(x)$.

A classic result of Faber [4] states that for *any* matrix X, there exists $f \in C[-1, 1]$ such that

$$\lim_{n \to \infty} \|H_{0,n}(X, f, x) - f(x)\| \neq 0,$$

where $\|f(x)\|$ denotes the supremum norm on $[-1, 1]$. On the other hand, if T denotes the matrix of Chebyshev nodes

$$T = \left\{ x_{k,n} = \cos\left(\frac{(2k-1)\pi}{2n}\right) : k = 1, 2, ..., n;\ n = 1, 2, 3, ... \right\},$$

and if the modulus of continuity $\omega(\delta) = \omega(\delta; f)$ of f is defined by

$$\omega(\delta) = \omega(\delta; f) = \sup\{|f(x) - f(y)| : x, y \in [-1, 1], |x - y| \leqslant \delta\},$$

then there exists a number c (independent of f and n) such that

$$\|H_{0,n}(T, f, x) - f(x)\| \leqslant c \log n\, \omega(1/n) \tag{1.1}$$

for all $f \in C[-1, 1]$ and $n \geqslant 2$. (See, for example, Szabados and Vértesi [14, Chap. 1].) Thus Lagrange polynomials based on the Chebyshev nodes T converge uniformly to f under the relatively mild condition $\omega(1/n) = o((\log n)^{-1})$, and so T provides a good choice of a node system for Lagrange interpolation. With regard to pointwise error estimates for Lagrange interpolation on T, Kis [7] showed there exists a number k_0, independent of f, n and x, so that

$$|H_{0,n}(T, f, x) - f(x)| \leqslant k_0 \left[\log n\, \omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i}\, \omega\left(\frac{i}{n^2}\right) \right] \tag{1.2}$$

for all $f \in C[-1, 1]$, $n \geqslant 2$ and $x \in [-1, 1]$. Note that (1.1) follows from (1.2) by $\omega(\sqrt{1-x^2}/n) \leqslant \omega(1/n)$ and

$$\omega\left(\frac{i}{n^2}\right) = \omega\left(\frac{i}{n} \times \frac{1}{n}\right) \leqslant \left(\frac{i}{n} + 1\right) \omega\left(\frac{1}{n}\right).$$

The study of higher-order HF interpolation is motivated by the famous result of Fejér [5] that if $f \in C[-1, 1]$, then $\|H_{1,n}(T, f, x) - f(x)\| \to 0$ as $n \to \infty$. A discussion of error estimates (both uniform and pointwise) for $(0, 1)$ HF interpolation on T is presented in Goodenough and Mills [6].

For (0, 1, 2) HF interpolation, Szabados and Varma [13] showed that for any matrix X, $H_{2,n}(X, f, x)$ cannot converge uniformly to $f(x)$ for all $f \in C[-1, 1]$. This result was generalized to $(0, 1, ..., 2m)$ HF interpolation for any m by Szabados [12], whose results showed that for any X, there exists $f \in C[-1, 1]$ such that

$$\lim_{n \to \infty} \|H_{2m,n}(X, f, x) - f(x)\| \neq 0.$$

A pointwise error estimate for (0, 1, 2) HF interpolation on the Chebyshev nodes was obtained by Byrne *et al.* [1], who showed that there is a number k_1, independent of f, n and x, so that

$$|H_{2,n}(T, f, x) - f(x)| \leqslant k_1 \left[(T_n(x))^2 \left(\log n\, \omega\left(\frac{\sqrt{1-x^2}}{n} \right) + \sum_{i=1}^{n} \frac{1}{i}\, \omega\left(\frac{i}{n^2} \right) \right) + \omega\left(\frac{|T_n(x)|}{n} \right) \right] \tag{1.3}$$

for all $f \in C[-1, 1]$, $n \geqslant 2$ and $x \in [-1, 1]$. Here $T_n(x)$ denotes the nth Chebyshev polynomial of the first kind,

$$T_n(x) = \cos(n \arccos x), \qquad -1 \leqslant x \leqslant 1,$$

whose zeros are $\cos((2k-1)\,\pi/(2n))$, $1 \leqslant k \leqslant n$. Since the right-hand side of (1.3) vanishes at the zeros of $T_n(x)$, the error estimate (1.3) reflects the fact that $H_{2,n}(T, f, x)$ interpolates $f(x)$ at these zeros. Further, since $|T_n(x)| \leqslant 1$ on $[-1, 1]$, it follows, as with Lagrange interpolation on the Chebyshev nodes, that the polynomials $H_{2,n}(T, f, x)$ converge uniformly to $f(x)$ on $[-1, 1]$ if $\omega(1/n) = o((\log n)^{-1})$.

The first aim of this paper is to generalize and sharpen (1.2) and (1.3) to HF interpolation of arbitrary even order on T. The following result will be established in Section 3.

THEOREM 1. *Suppose $f \in C[-1, 1]$. Then, for $m \geqslant 0$, $n \geqslant 2$ and $x \in [-1, 1]$,*

$$|H_{2m,n}(T, f, x) - f(x)| = O(1) \left[|T_n(x)|^{2m+1} \left(\log n\, \omega\left(\frac{\sqrt{1-x^2}}{n} \right) + \sum_{i=1}^{n} \frac{1}{i}\, \omega\left(\frac{i}{n^2} \right) \right) + \omega\left(\frac{|T_n(x)|}{n} \right) \right], \tag{1.4}$$

where the $O(1)$ term depends only on m.

We remark that for Lagrange interpolation, Kis' result (1.2) is a consequence of our Theorem 1. To see this, use $|T_n(x)| \leqslant 1$ in (1.4), together with the result

$$\omega\left(\frac{1}{n}\right) = \sum_{i=1}^{n} \frac{1}{n}\,\omega\left(\frac{n}{i} \times \frac{i}{n^2}\right)$$
$$\leqslant \sum_{i=1}^{n} \frac{1}{n}\left(\frac{n}{i}+1\right)\omega\left(\frac{i}{n^2}\right) \leqslant 2 \sum_{i=1}^{n} \frac{1}{i}\,\omega\left(\frac{i}{n^2}\right). \tag{1.5}$$

Also note that for $m=1$, (1.4) is a slightly stronger result than (1.3) because it incorporates an additional factor of $|T_n(x)|$ into part of the right-hand side. Finally, observe that from (1.4) it follows that for any $m \geqslant 0$, the polynomials $H_{2m,n}(T, f, x)$ converge uniformly to $f(x)$ whenever $\omega(1/n) = o((\log n)^{-1})$.

Our second aim in this paper is to study the equiconvergence behaviour of Lagrange and $(0, 1, ..., 2m)$ HF interpolation on the Chebyshev nodes. In this regard, G. Min (personal communication to T. M. Mills, 1994) showed that if $f \in C[-1, 1]$, then

$$\lim_{n \to \infty} (H_{2,n}(T, f, x) - f(x)) - \tfrac{1}{2}(T_n(x))^2 (H_{0,n}(T, f, x) - f(x)) = 0$$

uniformly for $-1 \leqslant x \leqslant 1$. Now, for $m = 1, 2, 3, ...,$ let

$$a_m = \frac{(2m)!}{2^{2m}(m!)^2}. \tag{1.6}$$

The following extension and generalization of Min's result will be proved in Section 4.

THEOREM 2. *Suppose $f \in C[-1, 1]$. Then, for $m \geqslant 1$ and $x \in [-1, 1]$,*

$$|(H_{2m,n}(T, f, x) - f(x)) - a_m (T_n(x))^{2m} (H_{0,n}(T, f, x) - f(x))|$$
$$= O(1)\left[\,|T_n(x)|^{2m+1}\left(\omega\left(\frac{\sqrt{1-x^2}}{n}\right) + \sum_{i=1}^{n} \frac{1}{i^2}\,\omega\left(\frac{i}{n^2}\right)\right) + \omega\left(\frac{|T_n(x)|}{n}\right)\right], \tag{1.7}$$

where the $O(1)$ term is dependent only on m.

Observe that the right-hand side of (1.7) is $O(1)\,\omega(1/n)$, and so $H_{2m,n}(T, f, x) \to f(x)$ as $n \to \infty$ if and only if $(T_n(x))^{2m}(H_{0,n}(T, f, x) - f(x)) \to 0$, where this can be interpreted in either the pointwise or uniform sense. We conclude, again in either the pointwise or uniform sense, that if $\lim_{n\to\infty} H_{0,n}(T, f, x) = f(x)$, then $\lim_{n\to\infty} H_{2m,n}(T, f, x) = f(x)$. For the converse, note that if $x = \cos(p\pi/q)$, where p, q are integers, and if $T_n(x) \neq 0$, then

$$|T_n(x)| = \left|\cos\left(\frac{np\pi}{q}\right)\right| = \left|\sin\left(\frac{(q-2np)\,\pi}{2q}\right)\right| \geqslant \left|\sin\left(\frac{\pi}{2q}\right)\right| \geqslant \frac{1}{q}.$$

Thus, by (1.7), if $x = \cos(p\pi/q)$ where p, q are integers, and if $\lim_{n\to\infty} H_{2m,n}(T, f, x) = f(x)$, then $\lim_{n\to\infty} H_{0,n}(T, f, x) = f(x)$. It seems to be an open question as to whether this result can be extended, either pointwise or uniformly, to all $x \in [-1, 1]$.

Theorem 2 has a second interpretation, in terms of a rational interpolatory method. Define

$$W_{m,n}(x) = 1 - a_m (T_n(x))^{2m},$$

and note that $a_m \leqslant 1/2$ for $m \geqslant 1$, so that $W_{m,n}(x) \geqslant 1/2$ for $-1 \leqslant x \leqslant 1$. Now, given $f \in C[-1, 1]$, define the rational function $R_{m,n}(f, x)$ by

$$R_{m,n}(f, x) = \frac{1}{W_{m,n}(x)}\,(H_{2m,n}(T, f, x) - a_m (T_n(x))^{2m}\, H_{0,n}(T, f, x)). \tag{1.8}$$

Observe that $R_{m,n}(f, x)$ has numerator of degree no greater than $(2m+1)\,n-1$ and denominator of degree $2mn$, and that $R_{m,n}(f, x)$ interpolates f at the Chebyshev nodes (which are the zeros of $T_n(x)$). For $m = 1, 2$, Xu [15] showed that there are constants c_m (independent of f, n and x) so that, if $x_k = \cos((2k-1)\,\pi/(2n))$, then

$$|R_{m,n}(f, x) - f(x)| \leqslant c_m \left(\frac{(T_n(x))^2}{n} \sum_{k=1}^{n} \left[\omega\left(\frac{\sqrt{1-x_k^2}}{n}\right) + \omega\left(\frac{1}{k^2}\right)\right] + \omega\left(\frac{|T_n(x)|}{n}\right)\right) \tag{1.9}$$

for all $f \in C[-1, 1]$, $n \geqslant 1$ and $x \in [-1, 1]$. This estimate reflects the fact that $R_{m,n}(f, x)$ interpolates f at the zeros of $T_n(x)$. Also, since

$$\sum_{k=1}^{n} \left[\omega\left(\frac{\sqrt{1-x_k^2}}{n}\right) + \omega\left(\frac{1}{k^2}\right)\right] \leqslant \sum_{k=1}^{n} \left[\omega\left(\frac{1}{n}\right) + \left(\frac{n}{k^2}+1\right)\omega\left(\frac{1}{n}\right)\right] = O(n)\,\omega\left(\frac{1}{n}\right),$$

it follows from (1.9) that for $m=1, 2$,

$$\|R_{m,n}(f,x)-f(x)\| = O(1)\,\omega(1/n),$$

and so the divergent (0, 1, ..., $2m$) and (0) HF processes have been combined to give an interpolation method that converges for all $f\in C[-1,1]$. Note that work with rational interpolatory schemes of a similar nature has been carried out by Byrne *et al.* [1], Meir [9], and Xu [16].

Our result concerning the rational interpolatory operator defined by (1.8) is the following corollary, which is obtained simply by dividing through (1.7) by $W_{m,n}(x)$.

COROLLARY. *Suppose $f\in C[-1,1]$. Then, for $m\geqslant 1$ and $x\in[-1,1]$,*

$$|R_{m,n}(f,x)-f(x)| = O(1)\left[\,|T_n(x)|^{2m+1}\left(\omega\left(\frac{\sqrt{1-x^2}}{n}\right)+\sum_{i=1}^{n}\frac{1}{i^2}\,\omega\left(\frac{i}{n^2}\right)\right)+\omega\left(\frac{|T_n(x)|}{n}\right)\right], \tag{1.10}$$

where the $O(1)$ term is dependent only on m.

Observe that (1.10) shows the approximation error vanishes at the nodes of interpolation, and also demonstrates that

$$\|R_{m,n}(f,x)-f(x)\| = O(1)\,\omega(1/n).$$

Thus, for each m, the rational interpolatory scheme defined by (1.8) combines the divergent (0, 1, ..., $2m$) HF and Lagrange methods on the Chebyshev nodes to form a new interpolatory process that converges uniformly for all $f\in C[-1,1]$.

2. PRELIMINARY RESULTS

In this section we collect together some results, mostly of a technical nature, that will be needed for the proofs of the theorems in Sections 3 and 4. Our main result is Theorem 3, which plays a key role in the proofs of Theorems 1 and 2 and is also of interest in its own right.

For an arbitrary interpolation matrix X, and f defined on $[-1,1]$, the (0, 1, ..., m) HF interpolation polynomial of f can be written as

$$H_{m,n}(X,f,x) = \sum_{k=1}^{n} f(x_{k,n})\,A_{k,m,n}(X,x), \tag{2.1}$$

where $A_{k,m,n}(X, x)$ is the unique polynomial of degree at most $(m+1)n-1$ such that

$$A_{k,m,n}^{(r)}(X, x_{j,n}) = \delta_{0,r}\,\delta_{k,j}, \qquad 1 \leqslant k, j \leqslant n, 0 \leqslant r \leqslant m.$$

(The $A_{k,m,n}(X, x)$ are referred to as the fundamental polynomials for $(0, 1, ..., m)$ HF interpolation on X.) We will henceforth be concerned solely with the Chebyshev nodes, and so for the remainder of this paper we adopt the shortened notation $H_{m,n}(f, x) = H_{m,n}(T, f, x)$, $A_{k,m,n}(x) = A_{k,m,n}(T, x)$, $\theta_k = \theta_{k,n} = (2k-1)\pi/(2n)$ and $x_k = x_{k,n} = \cos\theta_k$. In the first two lemmas we develop a useful representation formula for the fundamental polynomials $A_{k,2m,n}(x)$.

LEMMA 1. *For $r = 0, 1, 2, ...,$ there exist positive constants $b_{p,r}$, $0 \leqslant p \leqslant r$, so that*

$$\frac{d^{2r}}{dx^{2r}}(\cot x) = \sum_{p=0}^{r} b_{p,r} \cot^{2p+1} x. \tag{2.2}$$

Proof. We use induction. If $r = 0$, the lemma is clearly true, and if it holds for $r = s$, then

$$\begin{aligned}
\frac{d^{2s+2}}{dx^{2s+2}}(\cot x) &= \frac{d^2}{dx^2}\left(\sum_{p=0}^{s} b_{p,s} \cot^{2p+1} x\right) \\
&= -\frac{d}{dx}\left(\sum_{p=0}^{s} b_{p,s}(2p+1)(\cot^{2p} x + \cot^{2p+2} x)\right) \\
&= \sum_{p=0}^{s} b_{p,s}(2p(2p+1)\cot^{2p-1} x + 2(2p+1)^2 \cot^{2p+1} x \\
&\qquad + (2p+1)(2p+2)\cot^{2p+3} x) \\
&= \sum_{p=0}^{s+1} b_{p,s+1} \cot^{2p+1} x,
\end{aligned}$$

say, where $b_{p,s+1} > 0$ for $0 \leqslant p \leqslant s+1$. ∎

For $m \geqslant 0$, define the positive constants $a_{r,m}$ to be the coefficients in the Laurent expansion

$$\csc^{2m+1}\theta = \frac{1}{\theta^{2m+1}} \sum_{r=0}^{\infty} a_{r,m}\theta^{2r}, \qquad 0 < |\theta| < \pi. \tag{2.3}$$

Note that $a_{m,m} = a_m$ (where a_m is defined by (1.6)). This is shown (using contour integration) in Byrne *et al.* [3], and can also be demonstrated by equating coefficients of θ^{-1} in the Laurent expansions about 0 of both sides of the identity

$$\csc^{2n+1}\theta - \frac{2n-1}{2n}\csc^{2n-1}\theta = -\frac{1}{2n}\frac{d}{d\theta}(\cot\theta\csc^{2n-1}\theta).$$

The recurrence relation

$$a_{n,n} - \frac{2n-1}{2n}a_{n-1,n-1} = 0$$

is obtained, from which the explicit formula for $a_{m,m}$ follows immediately.

LEMMA 2. *The fundamental polynomials for* $(0, 1, ..., 2m)$ *HF interpolation on* T *can be written as*

$$A_{k,2m,n}(\cos\theta) = (-1)^{k-1}\cos^{2m+1} n\theta \times \sum_{r=0}^{m}\sum_{p=0}^{r}\frac{a_{m-r,m}b_{p,r}}{(2n)^{2r+1}(2r)!}\left[\cot^{2p+1}\frac{1}{2}(\theta+\theta_k) - \cot^{2p+1}\frac{1}{2}(\theta-\theta_k)\right], \tag{2.4}$$

where the $a_{m-r,m}$ *are defined by* (2.3) *and the* $b_{p,r}$ *are given by* (2.2).

Proof. As shown in the proof of Theorem 1 of Byrne *et al.* [3],

$$A_{k,2m,n}(\cos\theta) = S_{2m,n}(\theta+\theta_k) + S_{2m,n}(\theta-\theta_k),$$

where

$$S_{2m,n}(\theta) = \frac{1}{2}\sin^{2m+1} n\theta \sum_{r=0}^{m}\frac{a_{m-r,m}}{n^{2r+1}(2r)!}\frac{d^{2r}}{d\theta^{2r}}\cot\frac{\theta}{2}.$$

(The derivation of this result relies on work of Kreß [8].) The lemma is then established by substituting (2.2) into this representation of $A_{k,2m,n}(\cos\theta)$. ∎

Next, for each $x \in [-1, 1]$, write $x = \cos\theta$, where $0 \leqslant \theta \leqslant \pi$, and choose j such that

$$\min\{|\theta_k - \theta| : 1 \leqslant k \leqslant n\} = |\theta_j - \theta|. \tag{2.5}$$

The following three lemmas are based on results of Kis [7].

LEMMA 3. *If j is defined by* (2.5) *and $f \in C[-1, 1]$, then*

$$|f(x_k) - f(x)| = \begin{cases} O(1)\, \omega\left(\dfrac{|\cos n\theta|}{n}\right), & \text{if } \quad k = j, \\ O(1)\left[\omega\left(\dfrac{i \sin \theta}{n}\right) + \omega\left(\dfrac{i^2}{n^2}\right)\right], & \text{if } \quad i = |k-j| \geqslant 1, \end{cases} \tag{2.6}$$

where the $O(1)$ terms are independent of all variables.

Proof. Suppose $k = j$. By Goodenough and Mills [6, Lemma 3],

$$|\theta_j - \theta| \leqslant \frac{\pi}{2n}\, |\cos n\theta|,$$

and so

$$|f(x_j) - f(x)| \leqslant \omega(|x_j - x|) \leqslant \omega(|\theta_j - \theta|) \leqslant \omega\left(\frac{\pi}{2n}\, |\cos n\theta|\right)$$
$$= O(1)\, \omega\left(\frac{|\cos n\theta|}{n}\right).$$

The case $i = |k - j| \geqslant 1$ of (2.6) is due to Kis [7, Lemma 1]. ▮

LEMMA 4. *If j is defined by* (2.5) *and $f \in C[-1, 1]$, then*

$$\begin{cases} |f(x_{k+1}) - f(x_k)| = O(1)\left[\omega\left(\dfrac{\sin \theta}{n}\right) + \omega\left(\dfrac{i}{n^2}\right)\right], & \text{if } \quad i = k - j \geqslant 1, \\ |f(x_k) - f(x_{k-1})| = O(1)\left[\omega\left(\dfrac{\sin \theta}{n}\right) + \omega\left(\dfrac{i}{n^2}\right)\right], & \text{if } \quad i = j - k \geqslant 1, \end{cases} \tag{2.7}$$

where the $O(1)$ term is independent of all variables.

Proof. See Kis [7, Lemma 2]. ▮

LEMMA 5. *If j is defined by* (2.5), *then*

$$|A_{k,0,n}(x)| = \begin{cases} O(1), & \text{if } \quad k = j, \\ O(1)\, |T_n(x)|/i, & \text{if } \quad i = |k - j| \geqslant 1, \end{cases} \tag{2.8}$$

$$\begin{cases} |A_{k,0,n}(x) + A_{k+1,0,n}(x)| = O(1)\, |T_n(x)|/i^2, & \text{if } \quad i = k - j \geqslant 1, \\ |A_{k,0,n}(x) + A_{k-1,0,n}(x)| = O(1)\, |T_n(x)|/i^2, & \text{if } \quad i = j - k \geqslant 1. \end{cases} \tag{2.9}$$

Here, the $O(1)$ terms are independent of all variables.

Proof. The statements (2.8) and (2.9) are established by Kis [7, Lemmas 3, 4], but without the factors of $|T_n(x)|$ on the right-hand sides. However, it is evident from the proofs given by Kis that the factor can be included as shown above. ∎

The following elementary inequality will be needed.

LEMMA 6. *If* $0 \leqslant \alpha, \beta \leqslant \pi$, *then*

$$\sin \tfrac{1}{2}(\alpha+\beta) \geqslant \sin \tfrac{1}{2}\, |\alpha-\beta|.$$

We aim now to generalize the results of Lemma 5 to the fundamental polynomials for HF interpolation of arbitrary even order. The following result, which is crucial to this goal, helps to explain quantitatively why the Lagrange and (0, 1, ..., $2m$) HF interpolation methods on T have similar approximation properties.

THEOREM 3. *If* a_m *and* j *are defined by* (1.6) *and* (2.5), *respectively, then for* $i=|k-j| \geqslant 1$,

$$A_{k,2m,n}(x)=a_m(T_n(x))^{2m}\, A_{k,0,n}(x)+O(1)\,\frac{|T_n(x)|^{2m+1}}{i^3}, \tag{2.10}$$

where the $O(1)$ *term depends only on* m.

Proof. By (2.4),

$$A_{k,2m,n}(x)=(-1)^{k-1}\,(T_n(x))^{2m+1} \sum_{r=0}^{m} \sum_{p=0}^{r} \frac{a_{m-r,m} b_{p,r}}{(2n)^{2r+1}\,(2r)!}$$
$$\times\left[\cot\frac{1}{2}(\theta+\theta_k)-\cot\frac{1}{2}(\theta-\theta_k)\right]$$
$$\times \sum_{q=0}^{2p} \cot^{2p-q}\frac{1}{2}(\theta+\theta_k)\,\cot^{q}\frac{1}{2}(\theta-\theta_k)$$

and

$$A_{k,0,n}(x)=(-1)^{k-1}\,\frac{T_n(x)}{2n}\left[\cot\frac{1}{2}(\theta+\theta_k)-\cot\frac{1}{2}(\theta-\theta_k)\right].$$

Thus,

$$A_{k,2m,n}(x)=A_{k,0,n}(x)(T_n(x))^{2m}\sum_{r=0}^{m}\sum_{p=0}^{r}\frac{a_{m-r,m}b_{p,r}}{(2n)^{2r}(2r)!}$$
$$\times\sum_{q=0}^{2p}\cot^{2p-q}\frac{1}{2}(\theta+\theta_k)\cot^{q}\frac{1}{2}(\theta-\theta_k)$$
$$=a_{m,m}(T_n(x))^{2m}A_{k,0,n}(x)+B_{k,2m,n}(x),$$

say, where

$$B_{k,2m,n}(x)=A_{k,0,n}(x)(T_n(x))^{2m}\sum_{r=1}^{m}\sum_{p=0}^{r}\frac{a_{m-r,m}b_{p,r}}{(2n)^{2r}(2r)!}$$
$$\times\sum_{q=0}^{2p}\cot^{2p-q}\frac{1}{2}(\theta+\theta_k)\cot^{q}\frac{1}{2}(\theta-\theta_k).$$

Now, by Lemma 6,

$$\left|\cot^{2p-q}\frac{1}{2}(\theta+\theta_k)\cot^{q}\frac{1}{2}(\theta-\theta_k)\right|\leqslant\frac{1}{|\sin^{2p-q}(1/2)(\theta+\theta_k)\sin^{q}(1/2)(\theta-\theta_k)|}$$
$$\leqslant\frac{1}{\sin^{2p}(1/2)(\theta-\theta_k)},$$

and if $i=|k-j|\geqslant 1$, then

$$\sin\frac{1}{2}|\theta-\theta_k|\geqslant\sin\left(\frac{(2i-1)\pi}{4n}\right)\geqslant\frac{2i-1}{2n}\geqslant\frac{i}{2n}.$$

Thus, by (2.8),

$$|B_{k,2m,n}(x)|=O(1)\frac{|T_n(x)|^{2m+1}}{i}$$
$$\times\sum_{r=1}^{m}\sum_{p=0}^{r}\frac{(2p+1)a_{m-r,m}b_{p,r}}{(2n)^{2r-2p}i^{2p}(2r)!}=O(1)\frac{|T_n(x)|^{2m+1}}{i^3},$$

which establishes (2.10). ∎

From Theorem 3, the following generalization of Lemma 5 is obtained.

LEMMA 7. *If j is defined by* (2.5), *then*

$$|A_{k,2m,n}(x)| = \begin{cases} O(1), & \text{if} \quad k=j, \\ O(1)\,|T_n(x)|^{2m+1}/i, & \text{if} \quad i=|k-j|\geqslant 1, \end{cases} \tag{2.11}$$

$$\begin{cases} |A_{k,2m,n}(x)+A_{k+1,2m,n}(x)| = O(1)\,|T_n(x)|^{2m+1}/i^2, & \text{if} \quad i=k-j\geqslant 1, \\ |A_{k,2m,n}(x)+A_{k-1,2m,n}(x)| = O(1)\,|T_n(x)|^{2m+1}/i^2, & \text{if} \quad i=j-k\geqslant 1. \end{cases} \tag{2.12}$$

Here, all $O(1)$ terms depend only on m.

Proof. The first part of (2.11) follows from Szabados [12, p. 367], where it is shown that $|A_{k,2m,n}(x)|$ is uniformly bounded for all k, n and x. (More precise results are given in Smith [11].) The remaining parts of the lemma follow immediately from (2.10) in conjunction with (2.8) and (2.9). ∎

Before concluding this section, we note that Theorem 3 has an interesting application to the Lebesgue constant $\Lambda_{2m,n}$ for $(0, 1, ..., 2m)$ HF interpolation on T, which is defined by

$$\Lambda_{2m,n} = \max_{-1\leqslant x\leqslant 1} \lambda_{2m,n}(x),$$

where

$$\lambda_{2m,n}(x) = \sum_{k=1}^{n} |A_{k,2m,n}(x)|.$$

It is known (see, for example, Rivlin [10, Sect. 1.3]) that

$$\Lambda_{0,n} = \lambda_{0,n}(\pm 1) = \frac{2}{\pi}\log n + O(1).$$

Now, by (2.10) and the first part of (2.11),

$$\begin{aligned} \lambda_{2m,n}(x) &= a_m (T_n(x))^{2m} \sum_{k=1}^{n} |A_{k,0,n}(x)| + O(1) \\ &\leqslant a_m \Lambda_{0,n} + O(1). \end{aligned}$$

On the other hand, also by (2.10) and the first part of (2.11),

$$\lambda_{2m,n}(\pm 1) = a_m \sum_{k=1}^{n} |A_{k,0,n}(\pm 1)| + O(1)$$
$$= a_m \Lambda_{0,n} + O(1).$$

Thus

$$\Lambda_{2m,n} = a_m \Lambda_{0,n} + O(1) = \frac{2}{\pi} \frac{(2m)!}{2^{2m}(m!)^2} \log n + O(1). \tag{2.13}$$

This result was obtained by Byrne *et al.* [2], with a sharper version being developed in [3]. However, both results relied on the characterization of $\Lambda_{2m,n}$ as $\lambda_{2m,n}(\pm 1)$, the proof of which is quite technical (see [2, pp. 351–357]). The above derivation of (2.13) does not depend on this characterization.

3. PROOF OF THEOREM 1

Our proof is based on a technique that was introduced by Kis [7], and which was also used by Byrne *et al.* [1]. Since $H_{2m,n}(1, x) \equiv 1$, it follows from (2.1) that

$$|H_{2m,n}(f, x) - f(x)| = \left| \sum_{k=1}^{n} (f(x_k) - f(x))\, A_{k,2m,n}(x) \right|.$$

For convenience, put

$$U_k = U_k(x) = (f(x_k) - f(x))\, A_{k,2m,n}(x), \tag{3.1}$$

so that

$$|H_{2m,n}(f, x) - f(x)| \leq \left| \sum_{k=1}^{j-1} U_k \right| + |U_j| + \left| \sum_{k=j+1}^{n} U_k \right|$$
$$= I_1 + I_2 + I_3, \qquad \text{say.} \tag{3.2}$$

(If j is 1 or n, then one of these terms is not present.)

We first estimate I_3. If $n-j$ is odd, then

$$I_3 \leqslant |U_{j+1}+U_{j+2}|+|U_{j+3}+U_{j+4}|+\cdots+|U_{n-2}+U_{n-1}|+|U_n|. \tag{3.3}$$

(If $n-j$ is even, the final term in the sum is $|U_{n-1}+U_n|$.) Now, for $k=j+i$, $1\leqslant i\leqslant n-j-1$, it follows from (3.1) that

$$\begin{aligned}|U_k+U_{k+1}| \leqslant{} & |f(x_k)-f(x)|\,|A_{k,\,2m,\,n}(x)+A_{k+1,\,2m,\,n}(x)| \\ & +|f(x_{k+1})-f(x_k)|\,|A_{k+1,\,2m,\,n}(x)|.\end{aligned}$$

Thus, by employing (2.6), (2.7), (2.11), and (2.12), and the property $\omega(i\delta)\leqslant i\omega(\delta)$ of the modulus of continuity, we have

$$|U_k+U_{k+1}| = O(1)\,\frac{|T_n(x)|^{2m+1}}{i}\left(\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{i}{n^2}\right)\right). \tag{3.4}$$

(Here and subsequently, the $O(1)$ terms depend only on m).

We next need to estimate $|U_n|$. Now, by (2.6) and (2.11),

$$\begin{aligned}|U_n| &= |f(x_n)-f(x)|\;|A_{n,\,2m,\,n}(x)| \\ &= O(1)\,\frac{|T_n(x)|^{2m+1}}{n-j}\left(\omega\left(\frac{(n-j)\sin\theta}{n}\right)+\omega\left(\frac{(n-j)^2}{n^2}\right)\right) \\ &= O(1)\,|T_n(x)|^{2m+1}\left(\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{1}{n}\right)\right) \\ &= O(1)\,|T_n(x)|^{2m+1}\sum_{i=1}^{n}\frac{1}{i}\left(\omega\left(\frac{\sin\theta}{n}\right)+\omega\left(\frac{i}{n^2}\right)\right),\end{aligned} \tag{3.5}$$

where the last line follows by the method used in (1.5). Since $\cos\theta=x$, it follows from (3.3), (3.4), and (3.5) that

$$I_3 = O(1)\,|T_n(x)|^{2m+1}\left(\log n\,\omega\left(\frac{\sqrt{1-x^2}}{n}\right)+\sum_{i=1}^{n}\frac{1}{i}\,\omega\left(\frac{i}{n^2}\right)\right).$$

A similar estimate holds for I_1, and for I_2 we have (by (2.6) and (2.11)),

$$I_2 = |f(x_j)-f(x)|\;|A_{j,\,2m,\,n}(x)| = O(1)\,\omega(|T_n(x)|/n).$$

The proof of Theorem 1 is now completed by substituting the above results into (3.2).

4. PROOF OF THEOREM 2

Since $H_{2m,n}(1, x) \equiv H_{0,n}(1, x) \equiv 1$, it follows that

$$|(H_{2m,n}(f, x) - f(x)) - a_m (T_n(x))^{2m} (H_{0,n}(f, x) - f(x))| = \left| \sum_{k=1}^{n} V_k \right|,$$

where

$$V_k = V_k(x) = (A_{k,2m,n}(x) - a_m (T_n(x))^{2m} A_{k,0,n}(x))(f(x_k) - f(x)).$$

Thus

$$|(H_{2m,n}(f, x) - f(x)) - a_m (T_n(x))^{2m} (H_{0,n}(f, x) - f(x))|$$
$$\leqslant \sum_{k=1}^{j-1} |V_k| + |V_j| + \sum_{k=j+1}^{n} |V_k| = J_1 + J_2 + J_3, \qquad \text{say}, \qquad (4.1)$$

where j is defined by (2.5). (As before, if j is 1 or n, then one of the terms on the right-hand side of (4.1) is not present.)

We consider J_3. On writing $i = k - j$, $j + 1 \leqslant k \leqslant n$, it follows from (2.6) and (2.10) that

$$|V_k| = O(1) \frac{|T_n(x)|^{2m+1}}{i^3} \left(\omega \left(\frac{i \sin \theta}{n} \right) + \omega \left(\frac{i^2}{n^2} \right) \right),$$

where the $O(1)$ term depends only on m. Thus,

$$J_3 = O(1)\, |T_n(x)|^{2m+1} \left(\omega \left(\frac{\sin \theta}{n} \right) + \sum_{i=1}^{n} \frac{1}{i^2}\, \omega \left(\frac{i}{n^2} \right) \right).$$

A similar estimate holds for J_1, and for J_2 we have from (2.6) and (2.11),

$$J_2 = O(1)\, \omega \left(\frac{|\cos n\theta|}{n} \right).$$

The result (1.7) then follows by substituting the results for J_1, J_2, and J_3 into (4.1), and recalling that $x = \cos \theta$.

REFERENCES

1. G. J. Byrne, T. M. Mills, and S. J. Smith, On Hermite–Fejér type interpolation on the Chebyshev nodes, *Bull. Austral. Math. Soc.* **47** (1993), 13–24.
2. G. J. Byrne, T. M. Mills, and S. J. Smith, The Lebesgue constant for higher order Hermite–Fejér interpolation on the Chebyshev nodes, *J. Approx. Theory* **81** (1995), 347–367.
3. G. J. Byrne, T. M. Mills, and S. J. Smith, The Lebesgue function for generalized Hermite–Fejér interpolation on the Chebyshev nodes, *J. Austral. Math. Soc. Ser. B*, in press.
4. G. Faber, Über die interpolatorische Darstellung stetiger Funktionen, *Jahresber. Deutsch. Math.-Verein.* **23** (1914), 190–210.
5. L. Fejér, Über interpolation, *Göttinger Nachr.* (1916), 66–91.
6. S. J. Goodenough and T. M. Mills, A new estimate for the approximation of functions by Hermite–Fejér interpolation polynomials, *J. Approx. Theory* **31** (1981), 253–260.
7. O. Kis, Remarks on the rapidity of convergence of Lagrange interpolation, *Ann. Univ. Sci. Budapest* **11** (1968), 27–40. [In Russian]
8. R. Kreß, On general Hermite trigonometric interpolation, *Numer. Math.* **20** (1972), 125–138.
9. A. Meir, An interpolatory rational approximation, *Canad. Math. Bull.* **21** (1978), 197–200.
10. T. J. Rivlin, "Chebyshev Polynomials," 2nd. ed., Wiley, New York, 1990.
11. S. J. Smith, On the fundamental polynomials for Hermite–Fejér interpolation of Lagrange type on the Chebyshev nodes, *J. Inequal. Appl.* **3** (1999), 267–277.
12. J. Szabados, On the order of magnitude of fundamental polynomials of Hermite interpolation, *Acta Math. Hungar.* **61** (1993), 357–368.
13. J. Szabados and A. K. Varma, On (0, 1, 2) interpolation in uniform metric, *Proc. Amer. Math. Soc.* **109** (1990), 975–979.
14. J. Szabados and P. Vértesi, "Interpolation of Functions," World Scientific, Singapore, 1990.
15. Yuan Xu, Rational Hermite–Fejér interpolation polynomials, *J. Engrg. Math.* **2** (1985), 144–151. [In Chinese]
16. Yuan Xu, On rational Hermite–Fejér type interpolation, *in* "Progress in Approximation Theory" (P. Nevai, and A. Pinkus, Eds.), pp. 867–877, Academic Press, Boston, 1991.